Topology 48 (2009) 105–110

Contents lists available at ScienceDirect

Topology

journal homepage: www.elsevier.com/locate/top

Sectorial operators on Wiener algebras of analytic functions

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ARTICLE INFO

MSC:
46G20
46E50

Keywords:
Infinite-dimensional holomorphy
Wiener algebras
Sectorial operators

ABSTRACT

Quantized operators acting on approximable Wiener type algebras of analytic functions with infinitely many variables are researched. For such operators a sufficient condition of a sectorial property is established and a holomorphic calculus is constructed.

1. Introduction

The Wiener algebra of absolutely convergent Taylor series of a complex variable

$$x(\xi)=\sum_{n\in\mathbb{N}}c_n\xi^n \quad \text{with} \quad \|x\|_W=\sum_{n\in\mathbb{N}}|c_n|<\infty$$

first appeared in [1] and now has wide applications in the operator theory (see e.g. [2]). Its infinite-dimensional analogue for Hilbert spaces has been investigated in [3], where the authors used homogeneous Hilbert–Schmidt polynomials instead of power addends $c_n\xi^n$. In the given paper, a Banach infinite-dimensional generalization, called approximable Wiener algebras, is considered. Namely, the approximable polynomials, as power addends in Taylor series instead of the Hilbert–Schmidt polynomials, are used.

Quantized sectorial operators acting on approximable Wiener algebras are the main object of our research. We investigate the following problem: is the quantized operator sectorial if so is the initial operator? A positive solution is given in Theorem 3.4. In Theorem 4.1 an application to a holomorphic calculus of quantized sectorial operators is specified.

For infinite-dimensional holomorphy, we refer the reader to [4,5] and for the theory of sectorial operators to [6,7].

2. Approximable Wiener algebras of analytic functions

Let $(X,\|\cdot\|)$ be a Banach space with the normed dual $(X',\|\cdot\|)$, and let $\langle X\mid X'\rangle=\{\langle x\mid\xi\rangle : x\in X,\xi\in X'\}$ denote their dual form. Let $B_{X'}=\{\xi\in X' : \|\xi\|<1\}$ and $\bar{B}_{X'}=\{\xi\in X' : \|\xi\|\le 1\}$ be the open and closed dual unit balls respectively. We consider only complex spaces. The notation $\otimes^n X'$ specifies the n-folds tensor product of X', which is the linear span of all elements $\xi_1\otimes\cdots\otimes\xi_n$ with $\xi_i\in X'$. The symmetric tensor power, which is the linear span of all elements

$$\xi_1\odot\cdots\odot\xi_n:=\frac{1}{n!}\sum_s\xi_{s(1)}\otimes\cdots\otimes\xi_{s(n)}$$

with $s:\{1,\ldots,n\}\longmapsto\{s(1),\ldots,s(n)\}$ running over all n-elements' permutations, is denoted by $\odot^n X'$. By the polarization formula we have that $\odot^n X'=\operatorname{span}\{\otimes^n\xi : \xi\in X'\}$. We can do the same for the spaces $\otimes^n X$ and $\odot^n X$.

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doi:10.1016/j.top.2009.11.008

If $\otimes^n X'$ (resp., $\odot^n X'$) is endowed with the projective norm

$$\|\zeta\|_{\otimes_\pi^n X'} = \inf\left\{\sum_{j\in\mathbb{N}}\prod_{i=1}^n \|\xi_{ij}\| : \zeta = \sum_j \xi_{1j}\otimes\cdots\otimes\xi_{nj} \in \otimes^n X',\ \xi_{ij}\in X'\right\}$$

then the corresponding completion of $\otimes^n X'$ is denoted by $\otimes_\pi^n X'$ (resp., $\odot_\pi^n X'$). The projection $s_{X'}^n : \xi_1 \otimes \cdots \otimes \xi_n \longmapsto \xi_1 \odot \ldots \odot \xi_n$, acting from $\otimes_\pi^n X'$ onto $\odot_\pi^n X'$ has the norm equal to 1. It is well-known [4] that the topological isomorphism

$$(\odot_\pi^n X')' \simeq \mathcal{P}^n(X')$$

(generally, not isometric) holds, where $\mathcal{P}^n(X')$ is the space of all n-homogeneous continuous complex polynomials endowed with the uniform norm on $\bar{B}_{X'}$.

The symmetric tensor power $\odot^n X$ of X can be endowed with the injective norm

$$\|x\|_{\odot_\varepsilon^n X} = \sup\left\{\left|\sum_{j\in\mathbb{N}}\lambda_j\langle\xi \mid x_j\rangle^n\right| : \xi\in\bar{B}_{X'}\right\},\quad x = \sum_j \lambda_j(\otimes^n x_j)\in\odot^n X$$

where $x_j \in X, \lambda_j \in \mathbb{C}$. Let $\odot_\varepsilon^n X$ denote the corresponding completion. The space $\odot_\varepsilon^n X$ coincides with a subspace in $\mathcal{P}^n(X')$ of approximable n-homogeneous polynomials on X' which are weakly star continuous on the bounded sets and the embedding

$$\mathcal{P}_\varepsilon^n(X') = \odot_\varepsilon^n X \hookrightarrow \mathcal{P}^n(X')$$

is isometric (see e.g.[4, p.112], [8]). Let $\mathcal{P}_\varepsilon(X')$ be the algebra of all approximable polynomials on X' which are weakly star continuous on the bounded subsets of X'.

Any analytic function $x : B_{X'} \ni \xi \longmapsto x(\xi) \in \mathbb{C}$ of bounded type has a unique Taylor expansion

$$x(\xi) = \sum_{n\in\mathbb{Z}_+}\frac{d^n x(\xi)}{n!}$$

with the Fréchet derivations $d^n x \in \mathcal{P}^n(X')$ at zero, uniformly convergent on the ball $\lambda\bar{B}_{X'}$ for any $\lambda \in (0, 1)$. In the sequel, for simplicity we put $x(0) = 0$.

Definition 2.1. Following [3] (where the case of Hilbert spaces X has been only considered) the direct ℓ_1-sum

$$W_\varepsilon(B_{X'}) = \sum_{n\in\mathbb{N}}\mathcal{P}_\varepsilon^n(X') = \left\{x = \sum_{n\in\mathbb{N}}\frac{d^n x}{n!} : d^n x\in\mathcal{P}_\varepsilon^n(X')\right\}$$

consisting of analytic complex functions $x : B_{X'} \ni \xi \longmapsto x(\xi)$ of bounded type on the dual Banach ball $B_{X'}$ with the finite ℓ_1-norm

$$\|x\|_W = \sum_{n\in\mathbb{N}}\frac{\|d^n x\|}{n!},\qquad \|d^n x\| = \sup_{\xi\in\bar{B}_{X'}}|d^n x(\xi)|$$

is called an approximable Wiener algebra. Clearly, $\mathcal{P}_\varepsilon(X')$ is dense in $W_\varepsilon(B_{X'})$.

Let us check that the approximable Wiener algebra are well-defined. Firstly, each $x \in W_\varepsilon(B_{X'})$ is analytic in the ball $B_{X'}$, since for its radius of uniform convergence the following holds:

$$\left(\limsup_{n\to\infty}\sqrt[n]{\|d^n x\|/n!}\right)^{-1} \ge \left(\limsup_{n\to\infty}\sqrt[n]{\|x\|_W}\right)^{-1} = 1.$$

Secondly, $W_\varepsilon(B_{X'})$ is a multiplicative algebra, since for all $x, y \in W_\varepsilon(B_{X'})$

$$x(\xi)y(\xi) = \sum_n\sum_{k=0}^n\frac{d^k x(\xi)}{k!}\frac{d^{n-k}y(\xi)}{(n-k)!} = \sum_n\frac{d^n x(\xi)y(\xi)}{n!},\quad \xi\in B_{X'},$$

$$\|xy\|_W \le \sum_n\sum_{k=0}^n\frac{\|d^k x\|}{k!}\frac{\|d^{n-k}y\|}{(n-k)!} = \|x\|_W\|y\|_W.$$

Note that $W_\varepsilon(B_{X'})$ is a Banach algebra, since the ℓ_1-sum of Banach spaces $\mathcal{P}_\varepsilon^n(X')$ is complete.

3. Quantized sectorial operators

For Banach spaces X, Y, the space of linear bounded operators endowed with the uniform norm is denoted by $\mathscr{L}(X, Y)$ and $\mathscr{L}(X) = \mathscr{L}(X, Y)$ if $X = Y$. For given $T_i \in \mathscr{L}(X, Y)$ the tensor product $T_1 \otimes \cdots \otimes T_n$ defined as

$$(T_1 \otimes \cdots \otimes T_n)x_1 \otimes \cdots \otimes x_n = T_1x_1 \otimes \cdots \otimes T_nx_n,$$

where $x_i \in X$ satisfies the following property (see [8]): the subspace of symmetric tensors is invariant with respect to $T_1 \otimes \cdots \otimes T_n$, i.e.,

$$(T_1 \otimes \cdots \otimes T_n)x_1 \odot \ldots \odot x_n \in \odot^n Y.$$

That is, $T_1 \otimes \cdots \otimes T_n \in \mathscr{L}(\odot_\varepsilon^n X, \odot_\varepsilon^n Y)$ and

$$\|T_1 \otimes \cdots \otimes T_n\|_{\mathscr{L}(\odot_\varepsilon^n X, \odot_\varepsilon^n Y)} = \|T_1\| \ldots \|T_n\|$$

for any $T_i \in \mathscr{L}(X, Y)$. If all $T_i^{-1} \in \mathscr{L}(Y, X)$ then

$$(T_1 \otimes \cdots \otimes T_n)^{-1} = T_1^{-1} \otimes \cdots \otimes T_n^{-1}$$

in $\mathscr{L}(\odot_\varepsilon^n Y, \odot_\varepsilon^n X)$.

To any given Banach spaces $\{X_\imath, \|\cdot\|_\imath\}_{\imath=0,1}$ with the identical map $I_0: X_0 \rightleftarrows X_0$ and a contracted dense embedding $I_1 := I_0|_{X_1}: X_1 \looparrowright X_0$ we assign the pairs

$$\mathscr{P}_\imath^n := \mathscr{P}_\varepsilon^n(X_\imath'), \qquad W_\imath := W_\varepsilon(B_{X_\imath'}), \quad \imath = 0, 1.$$

Clearly, $I_0 I_1 = I_1$. Consider the unit $\otimes^n I_0$ in $\mathscr{L}(\mathscr{P}_0^n)$ and the contracted dense embedding $\otimes^n I_1: \mathscr{P}_1^n \looparrowright \mathscr{P}_0^n$. Given $A \in \mathscr{L}(X_1, X_0)$, we define in $\mathscr{L}(\mathscr{P}_1^n, \mathscr{P}_0^n)$ the operator

$$\Gamma({}^nA) := \sum_{j=1}^n \Gamma({}_j^nA), \qquad \Gamma({}_j^nA) := \otimes^{j-1} I_0 \otimes A \otimes^{n-j} I_0.$$

The Wiener algebras $W_\imath$ and the weighted direct ℓ_1-sum

$$W_0^+ = \left\{ x = \sum_{n\in\mathbb{N}} \frac{d^n x}{n!} : d^n x \in \mathscr{P}_\varepsilon^n(X_0'),\ \|x\|_{W_0^+} := \sum_{n\in\mathbb{N}} \frac{\|d^n x\|_{\odot_\varepsilon^n X_0}}{(n+1)!} < \infty \right\}$$

form a Gelfand triple with the contracted dense embeddings

$$W_1 \looparrowright W_0 \looparrowright W_0^+.$$

Denote $\mathrm{I}_\imath := \left[\begin{cases} \otimes^n I_\imath & : n = k \\ 0 & : n \neq k \end{cases}\right]_{n,k\in\mathbb{N}}$. Evidently, I_0 is the unit in both $\mathscr{L}(W_0)$ and $\mathscr{L}(W_0^+)$, and the dense embedding $\mathrm{I}_1: W_1 \looparrowright W_0$ is contracted.

Proposition 3.1. *For any operator $A \in \mathscr{L}(X_1, X_0)$ the matrix diagonal operator $\Gamma(A) := \left[\begin{cases} \Gamma({}^nA) & : n = k \\ 0 & : n \neq k \end{cases}\right]_{n,k\in\mathbb{N}}$ belongs to $\mathscr{L}(W_1, W_0^+)$ and*

$$\|\Gamma(A)\|_{\mathscr{L}(W_1, W_0^+)} \le \|A\|_{\mathscr{L}(X_1, X_0)}. \tag{1}$$

Proof. On the total subset of elements $x = \sum_n \otimes^n y/n! \in W_1$ with $y \in X_1$,

$$\Gamma(A)x = \sum_n \Gamma({}^nA)\frac{\otimes^n y}{n!} = \sum_n \sum_{1\le j\le n} \frac{\otimes^{j-1} y \otimes Ay \otimes^{n-j} y}{n!}.$$

It follows that

$$\begin{aligned} \|\Gamma({}^nA)(\otimes^n y)\|_{\odot_\varepsilon^n X_0} &\le \sum_{1\le j\le n} \|Ay\|_{X_0} \|y\|_{X_1}^{n-1} \\ &\le n\|A\|_{\mathscr{L}(X_1,X_0)} \|y\|_{X_1}^n = n\|A\|_{\mathscr{L}(X_1,X_0)} \|\otimes^n y\|_{\odot_\varepsilon^n X_1}, \end{aligned}$$

$$\|\Gamma(A)x\|_{W_0^+} \le \|A\|_{\mathscr{L}(X_1,X_0)} \sum_n \frac{n\|\otimes^n y\|_{\odot_\varepsilon^n X_1}}{(n+1)!} \le \|A\|_{\mathscr{L}(X_1,X_0)} \|x\|_{W_1}.$$

Hence, $\Gamma(A) \in \mathscr{L}(W_1, W_0^+)$ and (1) holds. □

Similarly as in the cases of Fock spaces [9], we call the operator $\Gamma(A) \in \mathscr{L}(W_1, W_0^+)$ a *quantization* of $A \in \mathscr{L}(X_1, X_0)$ on corresponding Wiener algebra.

Remark 3.2. The spaces $\mathscr{L}(X_1, X_0)$ can be treated as a set of unbounded linear operators on X_0 with the common dense domain X_1. Analogically, we can treat $\mathscr{L}(\mathscr{P}_1^n, \mathscr{P}_0^n)$ and $\mathscr{L}(W_1, W_0^+)$.

By $\varrho(A)=\{\lambda\in\mathbb{C}\colon(\lambda I_1-A)^{-1}\in\mathscr{L}(X_0,X_1)\}$ and $\sigma(A)=\mathbb{C}\setminus\varrho(A)$ we denote the resolvent set and the spectrum of an unbounded operator $A\in\mathscr{L}(X_1,X_0)$ on the space X_0. There is well-defined an analytic resolvent

$$\varrho(A)\ni\lambda\longmapsto R(\lambda,A):=I_1(\lambda I_1-A)^{-1}\in\mathscr{L}(X_0).$$

If $\varrho(A)\neq\varnothing$, then the operator $A\in\mathscr{L}(X_1,X_0)$ is closed on X_0. Then $\sigma(A)$ is closed and $\varrho(A)$ is open in $\mathbb{C}$. For operators in $\mathscr{L}(W_1,W_0^+)$ and $\mathscr{L}(\mathscr{P}_1^n,\mathscr{P}_0^n)$, the resolvent sets and spectra are defined analogously.

It is convenient to introduce a bit modified definition of sectorial operators, given in the book [6, ch. V]. Let $\Lambda(\vartheta)=\{re^{i\vartheta}\in\mathbb{C}\colon r\geq 0\}$ denotes a ray with fixed $\vartheta\in[0,2\pi]$ and $\Theta(\alpha)=\{\Lambda(\vartheta)\colon\vartheta\in[-\alpha,\alpha]\}\subset\mathbb{C}$ is a closed sector with fixed $\alpha\in(\pi/2,\pi)$. An operator $A\in\mathscr{L}(X_1,X_0)$ is called *sectorial* if there exists a sector $\Theta(\alpha)$ and constants $\delta=\delta(A)$, $K_\delta=K_\delta(A)$ such that

$$\sup_{\lambda\in\Theta_\delta^\alpha}\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0,X_1)}:=K_\delta,\qquad\Theta_\delta^\alpha:=\Theta(\alpha)\setminus\{\lambda\in\mathbb{C}\colon|\lambda|<\delta\}\subset\varrho(A).$$

The subset in $\mathscr{L}(X_1,X_0)$ of sectorial operators with the sector Θ_δ^α is denoted by $\Theta_\delta^\alpha(X_1,X_0)$. If $A\in\Theta_\delta^\alpha(X_1,X_0)$ then $\sigma(A)\subset\mathbb{C}\setminus\Theta_\delta^\alpha$. Similarly the set of sectorial operators on the pairs $\mathscr{P}_\imath^n$ can be defined.

Definition 3.3. We say that an operator $T\in\mathscr{L}(W_1,W_0^+)$ has the (weak) *sectorial property* if

$$\sup_{\lambda\in\Theta_\delta^\alpha}\|(\lambda\mathrm{I}_1-T)^{-1}\|_{\mathscr{L}(W_1)}<\infty,\quad\Theta_\delta^\alpha\subset\varrho(T).$$

The subset in $\mathscr{L}(W_1,W_0^+)$ of operators with these sectorial properties in the sector Θ_δ^α is denoted by $\Theta_\delta^\alpha(W_1)$. If $T\in\Theta_\delta^\alpha(W_1)$, then $\sigma(T)\subset\mathbb{C}\setminus\Theta_\delta^\alpha$.

It is known [6, Th 5.3] that any set of sectorial operators $\Theta_\delta^\alpha(X_1,X_0)$ is open in $\mathscr{L}(X_1,X_0)$. Similarly, it can be shown that $\Theta_\delta^\alpha(W_1)$ is open in $\mathscr{L}(W_1,W_0^+)$.

At once, it is easy to notice that

$$\varrho[\Gamma({}_j^nA)]=\varrho(A),\qquad[\lambda(\otimes^n I_1)-\Gamma({}_j^nA)]^{-1}=\otimes^{j-1}I_0\otimes(\lambda I_1-A)^{-1}\otimes^{n-j}I_0$$

with $1\leq j\leq n$ for all $A\in\mathscr{L}(X_1,X_0)$ and $\lambda\in\varrho(A)$. Therefore, for any $A\in\Theta_\delta^\alpha(X_1,X_0)$ we have

$$\sup_{\lambda\in\Theta_\delta^\alpha}\|[\lambda(\otimes^n I_1)-\Gamma({}_j^nA)]^{-1}\|_{\mathscr{L}(\mathscr{P}_0^n,\mathscr{P}_1^n)}=\sup_{\lambda\in\Theta_\delta^\alpha}\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0,X_1)}=K_\delta.$$

As a result, $\Gamma({}_j^nA)\in\Theta_\delta^\alpha(\mathscr{P}_1^n,\mathscr{P}_0^n)$.

A much more difficult question is: does the quantized operator $\Gamma(A)$ has a sectorial property if A is sectorial? A positive answer is given by the following theorem.

Theorem 3.4. *If $A\in\Theta_\delta^\alpha(X_1,X_0)$, then $\Gamma(A)\in\Theta_\gamma^\alpha(W_1)$ and the norms of corresponding resolvents have the following estimations*

$$\begin{aligned}&\sup_{\lambda\in\Theta_\delta^\alpha}\|\lambda R(\lambda,A)\|_{\mathscr{L}(X_0)}\leq 1+K_\delta\|A\|_{\mathscr{L}(X_1,X_0)}:=C_\delta,\\&\sup_{\lambda\in\Theta_\gamma^\alpha}\|\lambda R[\lambda,\Gamma({}^nA)]\|_{\mathscr{L}(\mathscr{P}_1^n)}\leq C_\delta,\\&\sup_{\lambda\in\Theta_\gamma^\alpha}\|\lambda R[\lambda,\Gamma(A)]\|_{\mathscr{L}(W_1)}\leq C_\delta\end{aligned}\tag{2}$$

with $\gamma=\max\{\delta,\ 4C_\delta\|A\|_{\mathscr{L}(X_1,X_0)}\}$, where $\Theta_\gamma^\alpha\subset\varrho[\Gamma(A)]\bigcap\Theta_\delta^\alpha$.

Proof. By the identity $\lambda I_1(\lambda I_1-A)^{-1}=I_0+A(\lambda I_1-A)^{-1}$ with $\lambda\in\varrho(A)$ and the inequality $\|A(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0)}\leq\|A\|_{\mathscr{L}(X_1,X_0)}\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0,X_1)}$, we obtain $\|\lambda I_1(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0)}\leq C_\delta$ with the constant $C_\delta=1+K_\delta\|A\|_{\mathscr{L}(X_1,X_0)}$ independent on $\lambda\in\Theta_\delta^\alpha$ and a given $\delta=\delta(A)>0$. Thus, the first inequality in (2) for $A\in\Theta_\delta^\alpha(X_1,X_0)$ holds.

Show that $\Gamma({}^nA)$ has a sectorial property on the space $\mathscr{P}_1^n$ with the sector Θ_γ^α. Use the relation

$$\Gamma({}^nA)=\otimes^{n-1}I_0\otimes A+\Gamma({}^{n-1}A)\otimes I_0,$$

defined on $\mathscr{P}_1^n$. The resolvent identity

$$\begin{aligned}&[\lambda(\otimes^n I_1)-\Gamma({}^nA)]^{-1}-[\lambda(\otimes^{n-1}I_1)-\Gamma({}^{n-1}A)]^{-1}\otimes I_0\\&\quad=[\lambda(\otimes^n I_1)-\Gamma({}^nA)]^{-1}[\otimes^{n-1}I_0\otimes A][\lambda(\otimes^{n-1}I_1)-\Gamma({}^{n-1}A)]^{-1}\otimes I_0\end{aligned}$$

for all $\lambda\in\varrho[\Gamma({}^nA)]\bigcap\varrho[\Gamma({}^{n-1}A)]$ implies that

$$\begin{aligned}&[\lambda(\otimes^n I_1)-\Gamma({}^nA)]^{-1}=\{[\lambda(\otimes^{n-1}I_1)-\Gamma({}^{n-1}A)]^{-1}\otimes I_0\}M[\lambda,\Gamma({}^nA)],\\&R[\lambda,\Gamma({}^nA)]=\{R[\lambda,\Gamma({}^{n-1}A)]\otimes I_0\}M[\lambda,\Gamma({}^nA)]\end{aligned}\tag{3}$$

with $M[\lambda,\Gamma(^nA)]=\{\otimes^n I_0-[\otimes^{n-1}I_0\otimes A][\lambda(\otimes^{n-1}I_1)-\Gamma(^{n-1}A)]^{-1}\otimes I_0\}^{-1}$ and $R[\lambda,\Gamma(^nA)]=(\otimes^n I_1)[\lambda(\otimes^n I_1)-\Gamma(^nA)]^{-1}$. Show that the right multiplier $M[\lambda,\Gamma(^nA)]$ in (3) can be expanded in a convergent power series. Note that

$$\|[\otimes^{n-1}I_0\otimes A][\lambda(\otimes^{n-1}I_1)-\Gamma(^{n-1}A)]^{-1}\otimes I_0\|^k_{\mathscr{L}(\mathscr{P}_1^n)}\le\|A\|^k_{\mathscr{L}(X_1,X_0)}\|R[\lambda,\Gamma(^{n-1}A)]\|^k_{\mathscr{L}(\mathscr{P}_1^n)}.$$

However, $\|R(\lambda,A)\|_{\mathscr{L}(X_1)}\le\|R(\lambda,A)\|_{\mathscr{L}(X_0)}\le C_\delta/|\lambda|$ for all $\lambda\in\Theta^\alpha_\delta$, in view of the first estimation (2). Using the fact that $\gamma=\max\{\delta,\ 4C_\delta\|A\|_{\mathscr{L}(X_1,X_0)}\}$ for all $\lambda\in\Theta^\alpha_\gamma\subset\Theta^\alpha_\delta$ and $n=2$ we have $\|(I_0\otimes A)(\lambda I_1-A)^{-1}\otimes I_0\|^k_{\mathscr{L}(\mathscr{P}_1^2)}\le\|A\|^k_{\mathscr{L}(X_1,X_0)}\|R(\lambda,A)\|^k_{\mathscr{L}(X_1)}\le 1/2^{k+1}$, because in this case

$$|\lambda|\ge 4C_\delta\|A\|_{\mathscr{L}(X_1,X_0)}.$$

Hence, $\|M[\lambda,\Gamma(^2A)]\|_{\mathscr{L}(\mathscr{P}_1^2)}\le 1$ and from (3) we obtain

$$\|[\lambda(\otimes^2 I_1)-\Gamma(^2A)]^{-1}\|_{\mathscr{L}(\mathscr{P}_1^2)}\le\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_1)}\le K_\delta,$$
$$\|\lambda R[\lambda,\Gamma(^2A)]\|_{\mathscr{L}(\mathscr{P}_1^2)}\le\|\lambda R(\lambda,A)\|_{\mathscr{L}(X_1)}\le C_\delta.$$

The induction by n implies that

$$\|[\lambda(\otimes^n I_1)-\Gamma(^nA)]^{-1}\|_{\mathscr{L}(\mathscr{P}_1^n)}\le\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_1)}\le K_\delta,$$
$$\|\lambda R[\lambda,\Gamma(^nA)]\|_{\mathscr{L}(\mathscr{P}_1^n)}\le\|\lambda R(\lambda,A)\|_{\mathscr{L}(X_1)}\le C_\delta$$

for all n and $\lambda\in\Theta^\alpha_\gamma$. Hence, $\Gamma(^nA)$ has a sectorial property on the space $\mathscr{P}_1^n$ and the second estimation in (2) holds.

Now the previous estimation we apply to the operators

$$[\lambda\mathrm{I}_1-\Gamma(^nA)]^{-1}=\left[\begin{cases}[\lambda(\otimes^n I_1)-\Gamma(^nA)]^{-1} & : n=k\\ 0 & : n\neq k\end{cases}\right]$$

and

$$R[\lambda,\Gamma(A)]=\left[\begin{cases}R[\lambda,\Gamma(^nA)] & : n=k\\ 0 & : n\neq k\end{cases}\right]$$

with $n,k\in\mathbb{N}$ defined on the Wiener algebras. On the total in W_1 subset of elements $x=\sum_n\otimes^n y/n!$ with $y\in X_1$ we have

$$\begin{aligned}\|[\lambda\mathrm{I}_1-\Gamma(A)]^{-1}x\|_{W_1}&\le\sum_{n\in\mathbb{N}}\frac{\|[\lambda(\otimes^n I_1)-\Gamma(^nA)]^{-1}(\otimes^n y)\|_{\odot^n_\varepsilon X_1}}{n!}\\&\le\|(\lambda I_1-A)^{-1}\|_{\mathscr{L}(X_0,X_1)}\|x\|_{W_1}\le K_\delta\|x\|_{W_1},\\ \|R[\lambda,\Gamma(A)]x\|_{W_1}&\le\|R(\lambda,A)\|_{\mathscr{L}(X_1)}\|x\|_{W_1}\le C_\delta\|x\|_{W_1}\end{aligned}$$

for all $\lambda\in\Theta^\alpha_\gamma$, hence $\Gamma(A)\in\Theta^\alpha_\gamma(W_1)$ and $\Theta^\alpha_\gamma\subset\varrho[\Gamma(A)]\bigcap\varrho(A)$. Thus, the third inequality in (2) holds.

Corollary 3.5. *If $A\in\Theta^\alpha_\delta(X_1,X_0)$, then*

$$\sigma[\Gamma(A)]\subset\mathbb{C}\setminus\Theta^\alpha_\gamma\quad\text{with}\quad\gamma=\max\{\delta,\ 4C_\delta\|A\|_{\mathscr{L}(X_1,X_0)}\}.$$

4. Holomorphic calculus for quantized sectorial operators

Let $A\in\Theta^\alpha_\gamma(X_1,X_0)$ and $\partial\Theta^\alpha_\gamma$ denote the boundary of the closed sector Θ^α_γ with a given angle $\alpha\in(\pi/2,\pi)$. Consider the space of scalar continuous functions $\mathscr{H}(\mathbb{C}\setminus\Theta^\alpha_\gamma)=\{\mathbb{C}\setminus\Theta^\alpha_\gamma\ni\lambda=r\mathrm{e}^{\vartheta}\longmapsto f(\lambda)\in\mathbb{C}\}$ analytic in the open complement $\mathbb{C}\setminus\Theta^\alpha_\gamma$ with the finite norm

$$\|f\|_\gamma=\frac{1}{\pi}\int_\gamma^\infty M_f(r)\frac{\mathrm{d}r}{r}+m_\gamma(f),\quad m_\gamma(f)=\sup_{\lambda\in\mathbb{C}\setminus\Theta^\alpha_\gamma}|f(\lambda)|,$$

where $M_f(r)=\max\{|f(r\mathrm{e}^{\mathrm{i}\vartheta})|:\vartheta\in[-\alpha,\alpha]\}$. It is easy to check that $\mathscr{H}(\mathbb{C}\setminus\Theta^\alpha_\gamma)$ is a Banach algebra for which

$$\|fg\|_\gamma\le\frac{m_\gamma(f)}{\pi}\int_\gamma^\infty M_g(r)\frac{\mathrm{d}r}{r}+m_\gamma(f)m_\gamma(g)=m_\gamma(f)\|g\|_\gamma\le\|f\|_\gamma\|g\|_\gamma$$

for all $f,g\in\mathscr{H}(\mathbb{C}\setminus\Theta^\alpha_\gamma)$, since $M_{fg}(r)\le M_f(r)M_g(r)$ and $M_f(r)\le m_\gamma(f)$. The convergence of improper integral implies that for all $f\in\mathscr{H}(\mathbb{C}\setminus\Theta^\alpha_\gamma)$ we have $\lim_{r\to\infty}M_f(r)=0$. Thus, $\mathscr{H}(\mathbb{C}\setminus\Theta^\alpha_\gamma)$ does not contain the unit.

Consider the contours

$$\Lambda_\gamma(\vartheta)=\{r\mathrm{e}^{\mathrm{i}\vartheta}:r\ge\gamma\},\qquad\Lambda^0_\gamma(\beta)=\{\gamma\mathrm{e}^{\mathrm{i}\vartheta}:\vartheta\in[\beta,2\pi-\beta]\}$$

with given $\beta\in(\pi/2,\alpha]$ and $\gamma>0$.

Theorem 4.1. *For any $A \in \Theta^\alpha_\delta(X_1, X_0)$ and $\gamma = \max\{\delta,\ 4C_\delta \|A\|_{\mathscr{L}(X_1,X_0)}\}$ the mappings*

$$\begin{aligned} \Phi \quad &: \mathscr{H}(\mathbb{C} \setminus \Theta^\alpha_\gamma) \ni f \longrightarrow f(A) \in \mathscr{L}(X_0), \\ \Gamma(\Phi) \quad &: \mathscr{H}(\mathbb{C} \setminus \Theta^\alpha_\gamma) \ni f \longrightarrow f[\Gamma(A)] \in \mathscr{L}(W_1) \end{aligned}$$

which are defined for the same functions f by the formulas

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Theta^\alpha_\gamma} f(\lambda) R(\lambda, A) \mathrm{d}\lambda,$$

$$f[\Gamma(A)] = \frac{1}{2\pi i} \int_{\partial\Theta^\alpha_\gamma} f(\lambda) R[\lambda, \Gamma(A)] \mathrm{d}\lambda,$$

respectively, are continuous homomorphisms of the corresponding algebras. The integration contours are positively oriented with respect to spectra and the integrals do not depend on choice of contours

$$\partial\Theta^\beta_\gamma = \Lambda_\gamma(-\beta) \bigcup \Lambda_\gamma(\beta) \bigcup \Lambda^0_\gamma(\beta) \subset \mathbb{C} \setminus \Theta^\alpha_\gamma$$

with any $\gamma \geq \delta$ and $\beta \in (\pi/2, \alpha]$.

Proof. The statement for the mapping Φ is well-known (e.g. [6, Ch. V]). Our result consists of the similar statement for $\Gamma(\Phi)$. This follows from the third estimation (2). Namely, for all $\lambda \in \Theta^\alpha_\gamma$ we have $\|R[\lambda, \Gamma(A)]\|_{\mathscr{L}(W_1)} \leq C_\delta |\lambda|$. If $\partial\Theta^\alpha_\gamma = \Lambda_\gamma(-\alpha) \bigcup \Lambda_\gamma(\alpha) \bigcup \Lambda^0_\gamma(\alpha)$ then for all $f \in \mathscr{H}(\mathbb{C} \setminus \Theta^\alpha_\gamma)$,

$$\begin{aligned} \|f[\Gamma(A)]\|_{\mathscr{L}(W_1)} &\leq \frac{C}{2\pi} \left[\int_\gamma^\infty |f(r\mathrm{e}^{-\mathrm{i}\alpha})| \frac{\mathrm{d}r}{r} + \int_\gamma^\infty |f(r\mathrm{e}^{\mathrm{i}\alpha})| \frac{\mathrm{d}r}{r} + \int_\alpha^{2\pi-\alpha} |f(r\mathrm{e}^{\mathrm{i}\vartheta})| \mathrm{d}\vartheta \right] \\ &\leq \frac{C}{\pi} \left[\int_\gamma^\infty M_f(r) \mathrm{d}r/r + (\pi - \alpha) m_\gamma(f) \right] \leq C \|f\|_\gamma, \end{aligned}$$

since $(\pi - \alpha)/\pi \leq 1/2$. The other properties of the mapping $\Gamma(\Phi)$ can be proved similarly as for Φ. $\square$

Corollary 4.2. *For any $A \in \Theta^\alpha_\delta(X_1, X_0)$ on the Wiener algebra W_1, an analytic operator semigroup*

$$z \longmapsto \mathrm{e}^{z\Gamma(A)} \in \mathscr{L}(W_1), \qquad \{z \in \mathbb{C} : |\arg(z)| < \alpha - \pi/2\}$$

with the generator $\Gamma(A)$ is well-defined.

Proof. The exponent $\mathrm{e}^{z\lambda} := e_z(\lambda)$ with $|\arg(z)| < \alpha - \pi/2$ belongs to the symbols algebra $\mathscr{H}(\mathbb{C} \setminus \Theta^\alpha_\gamma)$ by the variable $\lambda \in \mathbb{C} \setminus \Theta^\alpha_\gamma$ for each $\gamma > 0$ and $\alpha \in (\pi/2, \pi)$, since $z(r, \vartheta) := |z\lambda| \cos[\arg(z) + \vartheta] = \mathrm{Re}(z\lambda) < 0$ for such z and all $\vartheta \in (-\alpha, \alpha)$. Hence, $\|e_z\|_\gamma \leq 1 + \int_\gamma^\infty \mathrm{e}^{z(r,\alpha)} \mathrm{d}r/r\pi = 1 + \int_{-z(\gamma,\alpha)}^\infty \mathrm{e}^{-s} \mathrm{d}s/s\pi < \infty$. Moreover, the derivative $e_z'(\lambda) = \lambda e_z(\lambda)$ also belongs to $\mathscr{H}(\mathbb{C} \setminus \Theta^\alpha_\gamma)$, since $\|e_z'\|_\gamma \leq 1 + \int_{-z(\gamma,\alpha)}^\infty \mathrm{e}^{-s} \mathrm{d}s/\pi < \infty$. Finally, we have to apply Theorem 4.1 and the fact that

$$e_z'[\Gamma(A)]\,|_{z=0}\, x = \Gamma(A) e_z[\Gamma(A)]\,|_{z=0}\, x = \Gamma(A) x$$

for all $x \in W_1$. $\square$

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